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A Non-Steady Flow of Liquid in a Porous Pipe with Variable Permeability*

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INTRODUCTION

Consider a compressible fluid in a vertical pipe filled with porous medium whose permeability is a function $k(x)$, $0 < x < \infty$; here x is the height parameter. The piezometric head $u(x, t)$ and the height of the column of fluid $s(t)$ then satisfy a free boundary parabolic system; see (1.2)–(1.6) below. The initial pressure $g(x) - x$ and the velocity $l(t)$ across $x = 0$ are given functions. In the case $k \equiv 1$, this problem was solved by different methods by Friedman and Jensen [5] and by Torelli [7].

In Section 1 we establish the existence and uniqueness of a solution to the free boundary problem when $k(x)$ is smooth, and in Section 2 we extend these results to the case where $k(x)$ is piecewise constant.

Our main interest, however, is to assume that $k(x)$ is piecewise constant with values 1 or k, and find the limit behavior of the solution $(u, s) = (u^k, s^k)$ as $k \to \infty$.

In Section 3 we consider the case

$$k(x) = 1 \quad \text{if} \quad 0 < x < 1, \qquad k(x) = k \quad \text{if} \quad 1 < x < \infty.$$

We prove that $(u^k, s^k) \to (\tilde{u}, \tilde{s})$ where $(\tilde{u}, \tilde{s})$ is defined as follows:

Let w be the solution of

$$\begin{aligned} w_t - w_{xx} &= 0 && (0 < x < 1, t > 0),\\ w_x(0, t) &= -l(t) && (t > 0),\\ w(x, 0) &= g(x) && (0 < x < 1),\\ w_t + ww_x &= 0 && (x = 1, t > 0), \end{aligned}$$

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and let σ_1 be the last time that $w(1, t) \geqslant 1$ for all $t \leqslant \sigma_1$. Then

$$\begin{aligned} \tilde{u}(x, t) &= w(x, t) && \text{if } 0 < x < 1, \quad 0 < t < \sigma_1, \\ \tilde{s}(t) &= w(1, t) && \text{if } 0 < t < \sigma_1, \\ \tilde{u}(x, t) &= \tilde{u}(1, t) && \text{if } 1 < x < \tilde{s}(t). \end{aligned}$$

Next, let (v, s) be the solution of the hydraulic problem with $k \equiv 1$ for $t > \sigma_1$, with $v(x, \sigma_1) = \tilde{u}(x, \sigma_1)$, $0 \leqslant x \leqslant 1$, and let σ_2 be the last time that $s(t) \leqslant 1$ if $\sigma_1 < t \leqslant \sigma_2$. Then

$$\tilde{u}(x, t) = v(x, t), \qquad \tilde{s}(t) = s(t) \qquad \text{if } 0 < x < s(t), \quad \sigma_1 < t \leqslant \sigma_2.$$

For $t > \sigma_2$ we define $\tilde{u}, \tilde{s}$ as in the case $t > 0$, except that instead of the initial values $w(x, 0) = g(x)$ we now take $w(x, \sigma_2) = \tilde{u}(x, \sigma_2)$.

In this way we proceed step-by-step to construct $(\tilde{u}, \tilde{s})$.

In Section 4 we consider the case

$$k(x) = k \qquad \text{if } 0 < x < 1, \qquad k(x) = 1 \qquad \text{if } 1 < x < b.$$

We show that, as $k \to \infty$, (u^k, s^k) converges to $(\tilde{u}, \tilde{s})$ where $\tilde{u}(x, t)$ does not depend on x if $x < 1$ and, for $x > 1$, $(\tilde{u}, \tilde{s})$ solves the hydraulic problem with $k \equiv 1$, where the condition at $x = 1$ is given by

$$u_x = u_t.$$

Finally, in Section 5 we find the limit behavior of (u^k, s^k) as $k \to \infty$, in case

$$\begin{aligned} k(x) &= 1 && \text{if } 0 < x < 1 \quad \text{or if } \beta < x < b, \\ k(x) &= k && \text{if } 1 < x < \beta, \end{aligned}$$

where $1 < \beta < b$.

In [1, 2] Caffarelli and Friedman considered the analogous problem for a stationary flow in a porous 2-dimensional dam. The methods and results of these papers are entirely different from those of the present paper.

1. Existence of a Solution

Let $k(x)$ be a C^1 function in x, $0 \leqslant x < \infty$, satisfying

$$k(x) > 0 \qquad (0 \leqslant x < \infty).$$

Let $g(x)$ be a C^2 function for $0 \leqslant x \leqslant b$, satisfying

$$g(x) > x \qquad \text{if } 0 \leqslant x < b, \quad g(b) = b.$$

where b is a fixed positive number. Finally, let $l(t)$ be a C^1 function for $0 \leqslant t < \infty$, satisfying

$$l(t) > -1 \quad (0 \leqslant t < \infty), \qquad -l(0) = g'(0),$$

$$k(0) \int_0^t l(s)\, ds > \tfrac{1}{2}b^2 - b - \int_0^b g(x)\, dx \qquad \text{for all} \quad t > 0. \tag{1.1}$$

Consider the free boundary problem: find functions $u(x, t)$, $s(t)$ satisfying:

$$u_t - (ku_x)_x = 0 \qquad \text{if} \quad 0 < x < s(t), \quad t > 0, \tag{1.2}$$

$$u = s \qquad \text{if} \quad x = s(t), \quad t > 0, \tag{1.3}$$

$$ku_x = -\dot{s} \qquad \text{if} \quad x = s(t), \quad t > 0, \tag{1.4}$$

$$u(x, 0) = g(x) \qquad \text{if} \quad 0 < x < b, \tag{1.5}$$

$$u_x(0, t) = -l(t) \qquad \text{if} \quad t > 0. \tag{1.6}$$

For any $T > 0$, let

$$\Omega_T = \{(x, t); 0 < x < x < s(t), 0 < t < T\}.$$

By a solution of (1.2)–(1.6) we always mean a classical solution, that is, $s(t)$ is continuously differentiable for $t \geqslant 0$, and, for any $T > 0$, u and u_x are continuous in $\bar{\Omega}_T$ and (1.2)–(1.6) are satisfied.

For any $\gamma > 0$, let

$$k_\gamma = \inf_{x \geqslant \gamma} k(x). \tag{1.7}$$

We shall impose the following conditions:

$$s(t) \geqslant \gamma \qquad \text{for} \quad 0 \leqslant t \leqslant T, \tag{1.8}$$

$$k(x)\, g'(x) < k_\gamma \qquad \text{if} \quad 0 \leqslant x \leqslant b, \tag{1.9}$$

$$-k(0)\, l(t) < k_\gamma \qquad \text{if} \quad 0 \leqslant t \leqslant T, \tag{1.10}$$

and prove some a priori estimates for the solution u, s assuming that it exists for $t \leqslant T$.

LEMMA 1.1. *If* (1.8)–(1.10) *hold then, in* Ω_T,

$$ku_x \leqslant \max\{\max_{0 \leqslant t \leqslant T} (-k(0)\, l(t)), \max_{0 \leqslant x \leqslant b} k(x)\, g'(x)\}, \tag{1.11}$$

$$ku_x \geqslant \min\{\min_{0 \leqslant t \leqslant T} (-k(0)\, l(t)), \min_{0 \leqslant x \leqslant b} k(x)\, g'(x)\}. \tag{1.12}$$

Proof. Set $w = ku_x$. It is easily verified that w satisfies the parabolic equation

$$w_t - kw_{xx} = 0.$$

By differentiating the relation (1.3) and using (1.4) we get

$$w_x = (ku_x)_x = u_t = \frac{\dot{s}^2}{k} + \dot{s} \qquad \text{on} \quad x = s(t). \tag{1.13}$$

We claim that

$$w \text{ cannot take a negative minimum in } \bar{\Omega}_T \text{ on the free boundary } x = s(t). \tag{1.14}$$

Indeed, if w takes a negative minimum at $(s(t_0), t_0)$ then, by the strong maximum principle, $w_x < 0$ at this point and, therefore, (1.13) yields

$$\frac{\dot{s}^2}{k} + \dot{s} < 0 \qquad \text{at} \quad (s(t_0), t_0).$$

But this is impossible since

$$\dot{s} = -w > 0 \qquad \text{at} \quad (s(t_0), t_0).$$

Applying the maximum principle for w in Ω_T and using (1.14), the assertion (1.12) follows.

Similarly one can establish the assertion (1.11), provided one knows that

$$w \text{ cannot take a positive maximum on the free boundary.} \tag{1.15}$$

Suppose (1.15) is false. Then, for some $t_0 \in (0, T]$,

$$B \equiv \sup_{\Omega_T} w = w(s(t_0), t_0) > 0.$$

By the maximum principle we get $w_x(s(t_0), t_0) > 0$, so that, by (1.13),

$$\frac{\dot{s}^2}{k} + \dot{s} > 0 \qquad \text{at} \quad t = t_0 .$$

Since, however, $\dot{s}(t_0) = -w(s(t_0), t_0)) = -B$, we find that

$$\frac{B^2}{k(s(t_0))} - B > 0,$$

that is,

$$B > k(s(t_0)) \geqslant k_\gamma . \tag{1.16}$$

Consider the function

$$h(\sigma) = \sup_{\Omega_\sigma} w.$$

By (1.9), (1.10),

$$A \equiv \max\{\max_{0 \leqslant t \leqslant T} (-k(0)\, l(t)), \max_{0 \leqslant x \leqslant b} g'(x)\} < k_\gamma$$

and by (1.16),

$$h(T) > k_\gamma .$$

Since $h(t)$ is continuous and $h(0) \leqslant A$, there must exist a point $\sigma \in (0, T)$ such that

$$A < h(\sigma) < k_\gamma . \tag{1.17}$$

The first inequality in (1.17) implies that

$$\max w \text{ in } \bar{\Omega}_\sigma$$

is attained only on the free boundary. Repeating the argument that led to (1.16), with T replaced by σ, we then obtain the inequality

$$h(\sigma) > k_\gamma$$

which contradicts the second inequality in (1.17). This completes the proof of (1.15).

LEMMA 1.2. *If the assumptions* (1.8)–(1.10) *are replaced by the condition*

$$k'(x) \geqslant 0, \tag{1.18}$$

then (1.12) *is still valid and, further,*

$$u(x, t) > x \qquad \textit{in} \quad \Omega_T , \tag{1.19}$$

$$\dot{s}(t) > -k \qquad \textit{if} \quad 0 < t < T, \tag{1.20}$$

and

$$k u_x \leqslant \max\{ \sup_{0 \leqslant x < \infty} k(x), \max_{0 \leqslant t \leqslant T} (-k(0)\, l(t)), \max_{0 \leqslant x \leqslant b} k(x)\, g'(x)\}. \tag{1.21}$$

Proof. The proof of (1.12) does not use the assumptions (1.8)–(1.10). Consider next the function $z = u - x$. It satisfies

$$(k z_x)_x - z_t = -k' \leqslant 0 \qquad \text{in} \quad \Omega_T$$

and $z \geqslant 0$ on $x = 0$, $z_x \leqslant 0$ on $t = 0$, $z = 0$ on $x = s(t)$. By the maximum principle, $z > 0$ in Ω_T. Since, further $z = 0$ on $x = s(t)$, $z_x < 0$ on $x = s(t)$, that is, $k(u_x - 1) = -\dot{s} - k$ is negative; this establishes (1.19), (1.20). Finally, (1.21) follows by the maximum principle and (1.20).

THEOREM 1.3. *Assume that* (1.9), (1.10) *hold. Then there exists a unique solution of* (1.2)–(1.6) *for all* $0 < t < T_\gamma$ *where* T_γ *is the first time that* $s(t) = \gamma$; *if, in particular,* (1.9), (1.10) *hold with*

$$k_\gamma = k_0 = \inf_{x \geqslant 0} k(x),$$

then there exists a unique solution of (1.2)–(1.6) *for all* $t > 0$.

Proof. The existence and uniqueness of a local solution u, s can be established by the method of integral equations as in the case $k \equiv 1$ [5]. This method uses the fundamental solution of $(ku_x)_x - u_t$; in case $k \not\equiv 1$, this fundamental solution is constructed, for instance, in [4]. Next, Lemma 1.1 provides an a priori estimate on $u_x(x, t)$, which enables us to continue the solution step-by-step in time, as long as $s(t) > 0$. Thus, in order to complete the proof, we have to show that $s(t)$ remains positive for all t.

Suppose this is not true, and let $\bar{t}$ be such that $s(t) > 0$ if $t < \bar{t}$, $s(\bar{t}) = 0$. We then integrate the equation

$$(ku_x)_x = u_t$$

over $\Omega_{\bar{t}}$ and make use of the boundary conditions (1.3)–(1.6). After some simplification we find that

$$k(0) \int_0^{\bar{t}} l(s)\, ds$$

is equal to the right hand side of (1.1); this was assumed to be impossible.

Using Lemma 1.2 instead of Lemma 1.1, we get:

THEOREM 1.4. *Assume that* (1.18) *holds. Then there exists a unique solution of* (1.2)–(1.6) *for all* $t > 0$.

2. k IS PIECEWISE CONSTANT

We shall be interested in the case where $k(x)$ is piecewise constant. For the sake of simplicity and definiteness, we shall concentrate on the following three cases:

$$k(x) = 1 \quad \text{if} \quad 0 < x < 1, \qquad k(x) = k \quad \text{if} \quad x > 1 \tag{2.1}$$

where k is a number >1,

$$k(x) = k \quad \text{if} \quad 0 < x < 1, \qquad k(x) = 1 \quad \text{if} \quad x > 1, \tag{2.2}$$

and

$$\begin{aligned} k(x) &= 1 \qquad \text{if} \quad 0 < x < 1 \quad \text{or} \quad x > \beta, \\ k(x) &= k \qquad \text{if} \quad 1 < x < \beta, \end{aligned} \tag{2.3}$$

where $1 < \beta < b$. We shall refer to these cases, repectively, as the cases $(1, k)$, $(k, 1)$ and $(1, k, 1)$.

Consider first the case $(1, k)$. We approximate the piecewise constant function $k(x)$ by smooth functions $k_n(x)$ (with $k_n'(x) \geq 0$) and denote by u^n, s^n the corresponding solution of (1.2)–(1.6) established in Theorem 1.4. From the estimates in Lemma 1.2 we also have, for any $T > 0$,

$$| u_x^n(x, t)| \leqslant C \text{ in } \Omega_T^n = \{(x, t); 0 < x < s^n(t), 0 < t < T\}, \tag{2.4}$$

where C is a generic constant independent of n. It follows, by taking $x = s^n(t)$, that

$$| \dot{s}^n(t)| \leqslant C \tag{2.5}$$

and, by (1.13),

$$| u_t^n(s^n(t), t)| \leqslant C. \tag{2.6}$$

We claim that there is an $\epsilon_1 > 0$ independent of n such that

$$s^n(t) > \epsilon_1 \qquad \text{if} \quad 0 < t < T.$$

Indeed, otherwise let $\bar{t}$ be the first point where some $s^n(t)$ becomes equal to ϵ_1. Integrating

$$(k_n u_x^n)_x = u_t^n$$

over $0 < x < s^n(t), 0 < t < \bar{t}$, we get a contradiction to (1.1) if ϵ_1 is sufficiently small.

We choose the k_n so that $k_n(x) \leqslant k$, $k_n(x) = 1$ if $0 < x < 1 - 1/n$, $k_n(x) = k$ if $x > 1 + 1/n$. Then for any ϵ_0 sufficiently small, if $x \in \Omega_T^n$, $0 < x < 1 - \epsilon_0$, $n > 1/\epsilon_0$,

$$u_t^n - u_{xx}^n = 0.$$

If $\epsilon_0 < \epsilon_1$ then, since u^n is bounded by C,

$$| u_t^n(\epsilon_0, t)| \leqslant C.$$

Applying the maximum principle to u_t^n (using (2.6)) we find that

$$| u_t^n(x, t)| \leqslant C \text{ in } \Omega_T^n, \qquad x \geqslant \epsilon_0, \tag{2.7}$$

provided n is sufficiently large (depending on ϵ_0). Hence

$$| u_{xx}^n(x, t)| \leqslant C \qquad \text{if} \quad \epsilon_0 < x < 1, \quad (x, t) \in \Omega_T^n, \tag{2.8}$$

$$k\, | u_x^n(x, t)| \leqslant C \qquad \text{if} \quad x > 1, \quad (x, t) \in \Omega_T^n. \tag{2.9}$$

From (2.4), (2.7) we obtain a uniform Lipschitz condition on u^n in $\Omega_T^n \cap \{x > \epsilon_0\}$. In $\Omega_T^n \cap \{x < \epsilon_0\}$ the u^n are also Holder continuous (with exponent α and constant C independent of n), since $l(t)$ is in C^1. We can extend the u^n to $0 < x < \infty$, $0 < t < T$ so that they are still equicontinuous. Recalling (2.5) we deduce that there is a subsequence $(u^{n'}, s^{n'})$ (which we again denote by (u^n, s^n)) so that

$$u^n \to u \text{ uniformly in } 0 < x < \infty,\ 0 < t < T,$$
$$s^n \to s \text{ uniformly in } 0 < t < T,$$

and $u_x{}^n \to u_x$, $u_{xx}^n \to u_{xx}$, $u_t{}^n \to u_t$ in compact subsets of the domain $\{0 < x < s(t),\ 0 < t < T\}$ which does not intersect the line $x = 1$. It is also clear that at points of $x = s(t)$ where $s(t) \neq 1$, u and s satisfy (1.3), (1.4), and that u satisfies (1.5), (1.6).

From the estimate

$$|u_{xx}(x, t)| \leqslant C \qquad \text{if} \quad x \neq 1,$$

we also deduce that $u_x(1 + 0, t)$ and $u_x(1 - 0, t)$ exist. Since

$$\overline{\lim}\, |(k_x u_x{}^n)_x| = \overline{\lim}\, |u_t{}^n| \leqslant C \qquad \text{if} \quad x \neq 1,$$

we easily conclude that

$$ku_x(1 + 0, t) = u_x(1 - 0, t) \qquad \text{if} \quad (x, t) \in \Omega_T. \tag{2.10}$$

Consider in Ω_T the functions

$$v^1(x, t) = u(x, t) - u\left(1 + \frac{1 - x}{k^{1/2}}, t\right),$$

$$v^2(x, t) = u(x, t) + \frac{u}{k^{1/2}}\left(1 + \frac{1 - x}{k^{1/2}}, t\right)$$

for $1 < x < 1 + \delta$, δ small. It is easily seen that $kv_{xx}^i = v_t{}^i$. Since $v^1(1 + 0, t) = 0$, $v_x{}^2(1 + 0, t) = 0$, we deduce that $v^i(x, t)$ is C^∞ for $1 \leqslant x < 1 + \delta$. Hence also

$$u(x, t) \text{ is } C^\infty \text{ in } \Omega_T \cap \{x \geqslant 1\} \text{ and in } \Omega_T \cap \{x \leqslant 1\}.$$

By a solution u, s of (1.2)–(1.6) in case $k(x)$ is given by (2.1), we understand that, for any $T < \infty$, $s(t)$ is Lipschitz continuous for $0 \leqslant t \leqslant T$ and continuously differentiable whenever $s(t) \neq 1$; u is continuous in $\bar{\Omega}_T$; u_x is continuous in $\bar{\Omega}_T \cap \{x \leqslant 1\}$ and in $\bar{\Omega}_T \cap \{x \geqslant 1\}$ with a jump given by (2.10), and (1.2)–(1.6) hold whenever $x \neq 1$.

THEOREM 2.1. *In the case* $(1, k)$ *there exists a unique solution of* (1.2)–(1.6).

Proof. We have already proved existence. We shall prove uniqueness by the method of integral equations. Let N^n be the Neumann function for $(k_n u_x)_x - u_t$ in $x > 0$. The argument used to show that $u = \lim u^n$ is a solution of (1.2)–(1.6) can be used to show that, for some subsequence $n' \to \infty$, $N^{n'} \to N$ where N is the Neumann function for $(ku_x)_x - u_t$, with k given by (2.1); its spatial derivative has a jump, i.e.,

$$N_\xi(x, \xi, t)|_{\xi=1-0} = kN_\xi(x, \xi, t)|_{\xi=1+0}\,. \tag{2.11}$$

We can now express any possible solution u, s in terms of this Neumann function, noting that in view of the jump relations (2.10), (2.11), no integrals over $x = 1$ will appear. The formula for representing u is thus the same as in case k is a smooth function. We can therefore repeat the standard argument for deriving an integral equation for $u_x(s(t))$, noting that $s(t)$ is Lipschitz continuous. This gives uniqueness.

The estimates of Lemmas 1.1, 1.2 are valid for each $k = k_n$, $u = u^n$. Hence, they are also valid for u. In the next section we shall be interested in the case where $k \to \infty$. We shall then require the conditions:

$$\begin{aligned} g'(x) &= 0 \qquad \text{if} \quad 1 < x < b, \\ g'(x) &< 1 \qquad \text{if} \quad 0 < x < 1. \end{aligned} \tag{2.12}$$

Using the estimates of Lemma 1.1 we obtain:

COROLLARY 2.2. *In the case* $(1, k)$, *if* (2.12) *holds then the unique solution* $(u, s) = (u^k, s^k)$ *of* (1.2)–(1.6) *satisfies, for any* $T > 0$,

$$\begin{aligned} k \mid u_x^k(x, t)| &\leqslant C \qquad \textit{if} \quad 1 < x < s^k(t), \quad 0 < t < T, \\ \mid u_x^k(x, t)| &< C \qquad \textit{if} \quad 0 < x < \min(1, s^k(t)), \quad 0 < t < T, \end{aligned} \tag{2.13}$$

where C *is a constant independent of* k.

Consider next the case $(k, 1)$. The concept of a solution is defined as in the case $(1, k)$, except that the jump relation for u_x becomes

$$ku_x(1 - 0, t) = u_x(1 + 0, t). \tag{2.14}$$

We now require the conditions:

$$\begin{aligned} kg'(x) &< 1 \qquad \text{if} \quad 0 < x < 1, \\ g'(x) &< 1 \qquad \text{if} \quad 1 < x < b, \\ kl(t) &> -1 \qquad \text{if} \quad t > 0. \end{aligned} \tag{2.15}$$

THEOREM 2.3. *In the case* $(k, 1)$, *if* (2.15) *holds then there exists a unique solution of* (1.2)–(1.6).

Later on we shall assume that

$$\begin{aligned} g'(x) &= 0 \qquad \text{if} \quad 0 < x < 1, \\ g'(x) &< 1 \qquad \text{if} \quad 1 < x < b, \\ l(t) &\equiv 0. \end{aligned} \tag{2.16}$$

Using Lemma 1.1 we have:

COROLLARY 2.4. *In the case* $(k, 1)$ *if* (2.16) *holds then the solution* (u^k, s^k) *of* (1.2)–(1.6) *satisfies, for any* $T > 0$,

$$\begin{aligned} k \mid u_x^k(x, t)| &\leqslant C \qquad \textit{if} \quad 0 < x < \min(1, s^k(t)), \quad 0 < t < T, \\ | u_x^k(x, t)| &\leqslant C \qquad \textit{if} \quad 1 < x < s^k(t), 0 < t < T, \end{aligned} \tag{2.17}$$

where C *is a constant independent of* k.

Consider finally the case $(1, k, 1)$ and assume

$$\begin{aligned} g'(x) &< 1 \qquad \text{if} \quad 0 < x < 1, \quad \text{or} \quad \beta < x < b, \\ kg'(x) &< 1 \qquad \text{if} \quad 1 < x < \beta. \end{aligned} \tag{2.18}$$

THEOREM 2.5. *In the case* $(1, k, 1)$, *if* (2.18) *holds then there exists a unique solution of* (1.2)–(1.6).

The solution u, s is such that u_x satisfies the obvious jump relations at $x = 1$ and at $x = \beta$.

Assuming further that

$$\begin{aligned} g'(x) &= 0 \qquad \text{if} \quad 1 < x < \beta, \\ g'(x) &< 1 \qquad \text{if} \quad 0 < x < 1 \quad \text{or} \quad \beta < x < b, \end{aligned} \tag{2.19}$$

and using Lemma 1.1, we have:

COROLLARY 2.6. *In the case* $(1, k, 1)$, *if* (2.19) *holds then the solution* (u^k, s^k) *of* (1.2)–(1.6) *satisfies, for any* $T > 0$,

$$\begin{aligned} k \mid u_x^k(x, t)| &\leqslant C \qquad \textit{if} \quad 1 < x < \min(s^k(t), \beta), 0 < t < T, \\ | u_x^k(x, t)| &\leqslant C \qquad \textit{if} \quad 0 < x < \min(s^k(t), 1) \textit{ or if} \\ & \qquad\qquad\quad \beta < x < \max(s^k(t), \beta), 0 < t < T, \end{aligned} \tag{2.20}$$

where C *is a constant independent of* k.

3. The Case $(1, k)$, $k \to \infty$

Consider the problem:

$$
\begin{aligned}
w_t - w_{xx} &= 0 &&\text{if } 0 < x < 1, \quad t > 0,\\
w_x(0, t) &= -l(t) &&\text{if } t > 0,\\
w(x, 0) &= g(x) &&\text{if } 0 < x < 1,\\
w_x + w w_t &= 0 &&\text{if } x = 1, \quad t > 0.
\end{aligned}
\tag{3.1}
$$

The solution w is taken in the sense that it is continuous up to the boundary, w_x is continuous for $0 \leqslant x \leqslant 1$, $t > 0$, and w_t is continuous for $0 < x \leqslant 1$, $t > 0$.

THEOREM 3.1. *There exists a unique solution w of (3.1) for all $t < \tilde{t}$ where $\tilde{t} = \sup\{t; w(1, s) > 0 \text{ if } 0 \leqslant s \leqslant t\}$.*

Notice that since $g(1) > 0$, $\tilde{t}$ is either positive or $+\infty$.

Proof. For any $\delta > 0$, denote by L_δ^2 the space of measurable functions $\zeta(t)$, $0 \leqslant t \leqslant \delta$, with finite norm

$$\| \zeta \| = \left(\int_0^\delta \zeta^2(t)\, dt \right)^{1/2}.$$

Let $\Sigma_M = \{\zeta \in L_\delta^2; \| \zeta \| < M\}$ where δ, M are positive numbers to be determined below. Define a mapping $\zeta \to T\zeta$ for any $\zeta \in \Sigma_M$ by $(T\zeta)(t) = v(1, t)$ where v is the solution of

$$
\begin{aligned}
v_t - v_{xx} &= 0 &&\text{if } 0 < x < t, \quad 0 < t < \delta,\\
v(0, t) &= -l(t) &&\text{if } 0 < t < \delta,\\
v(x, 0) &= g'(x) &&\text{if } 0 < x < 1,\\
v(1, t) + \hat{\zeta}(t)\, v_x(1, t) &= 0 &&\text{if } 0 < t < \delta,
\end{aligned}
\tag{3.2}
$$

and $\hat{\zeta}$ is defined by

$$\hat{\zeta}(t) = \left(g^2(1) - 2 \int_0^t \zeta(1, s)\, ds \right)^{1/2}.$$

We choose $\delta < \frac{1}{4}$ and $M = \frac{1}{2} g^2(1)$. Then $\hat{\zeta}(t) > \frac{1}{2} g^2(1) > 0$ and therefore (3.2) has a unique solution. Notice that

$$| \hat{\zeta}(t) - \hat{\zeta}(t') | \leqslant C \, | t - t' |^{1/2}$$

where C is a generic constant independent of δ, ζ. By well known results on parabolic equations [4, Chap. 5] v satisfies

$$| v(x, t) - v(x', t')| \leqslant C(| x - x' |^{1/2} + | t - t' |^{1/2}), \qquad | v(x, t)| \leqslant C. \quad (3.3)$$

It follows that

$$\left(\int_0^\delta v^2(1, t)\, dt\right)^{1/2} \leqslant C\delta^{1/2} \leqslant M \qquad \text{if} \quad \delta < (M/C)^2.$$

But then T maps Σ_M into a compact subset. By Schauder's fixed point theorem, T has a fixed point ζ. Using the Schauder estimates [4] and a bootstrap argument, we find that the corresponding solution v (with $v(1, t) = \zeta(t)$) is C^∞ in $0 < x \leqslant 1$, $t > 0$; also, v_x is continuous if $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant \delta$ (by [4; p. 147]).

Now, the system of equations in (3.2) is formally obtained from (3.1) if we set $w_x = v$. Having solved (3.1), we now define w by

$$w(x, t) = \int_0^x v(y, t)\, dy + \int_0^t v_x(0, \tau)\, d\tau + g(0).$$

It is then easily verified that w is a solution of (3.1).

To prove uniqueness, suppose $\bar{w}$ is another solution and suppose $z \equiv w - \bar{w}$ takes a positive maximum at $(1, t_0)$. Then $(w^2 - \bar{w}^2)_t \geqslant 0$, $(w - \bar{w})_x < 0$ at $(1, t_0)$, which is impossible since $w_x + ww_t = \bar{w}_x + \bar{w}\bar{w}_t = 0$ at the points $(1, t)$. Since z cannot take a positive maximum elsewhere on the parabolic boundary, we conclude that $z \leqslant 0$. Similarly $z \geqslant 0$, so that $w = \bar{w}$ in their common domain of definition.

We can now proceed to prove global existence of w step-by-step. As long as $w(1, t)$ remains positive, we can continue the solution to a larger interval $0 < t < t + \delta$, where δ depends only on a positive lower bound for $w(1, t)$. The proof of Theorem 3.1 thereby follows.

LEMMA 3.2. *Let* $0 < \sigma < \bar{t}$. (a) *If there exists an interval* I: $1 - \delta < x < 1$, $t = \sigma$, *such that* $w_x(x, \sigma) < 0$ *on this interval, then* $w(1, t)$ *is strictly increasing in some interval* $\sigma < t < \sigma + \eta$.

(b) *If there exists an interval* I: $1 - \delta < x < 1$, $t = \sigma$ *such that* $w_x(x, \sigma) > 0$ *on this interval, then* $w(1, t)$ *is strictly decreasing in some interval* $\sigma < t < \sigma + \eta$.

Proof. To prove (a), construct a regular level curve γ: $w_x = -\epsilon$ given by $x = x(t)$, $\sigma < t < \sigma + \eta$ with $1 - \delta < x(\sigma) < 1$; this is possible by Sard's lemma. Denote by $\tilde{\gamma}$ the line segment $x = 1$, $\sigma < t < \sigma + \eta$, and denote by R the region bounded by $\tilde{\gamma}$, γ and the lines $t = \sigma$, $t = \sigma + \eta$.

We claim that $w_x < 0$ in $R \cup \tilde{\gamma}$. Indeed, otherwise w_x takes a nonnegative maximum at a point $(1, t^*) \in \tilde{\gamma}$. By the maximum principle

$$w_{xx}(1, t^*) > 0, \text{ that is, } w_t(1, t^*) > 0.$$

Since, however,

$$w_x(1, t^*) = -(ww_t)(1, t^*) \text{ is } \geqslant 0,$$

and since $w(1, t^*) > 0$, we arrive at a contradiction.

We have proved that

$$w_t(1, t) = -\left(\frac{w_x}{w}\right)(1, t) > 0 \qquad \text{on} \quad \tilde{\gamma},$$

that is, $w(1, t)$ is strictly increasing. This completes the proof of (a). The proof of (b) is similar.

We shall now impose the following restriction:

$l(t)$ and $g'(x)$ change sign at most a finite number N of times, that is, the number N_1 of points t_i where $l(t_i) = 0$ and the number N_2 of points x_i where $g'(x_i) = 0$ are both finite, and $N_1 + N_2 = N$. (3.4)

This condition implies that for any $t \in (0, \bar{t})$, $t \neq t_i$, there exist at most $[(N + 1)/2]$ disjoint intervals $(a_i(t), b_i(t))$ (with $b_i(t) < a_{i+1}(t)$) contained in $0 < x < 1$ such that $w_x(x, t) < 0$ if and only if x belongs to $(a_i(t), b_i(t))$ for some i and $w_x(x, t) > 0$ if and only if x belongs to $(b_{i+1}(t), a_i(t))$ for some i. In fact, this follows by using regular level curves $w_x = c$; for the hydraulic problem with $k \equiv 1$ this is done in [5], and for the present problem (3.1) the proof is the same. (Notice that if, for some t, $w_x(x, t) \equiv 0$ in an x-interval, then, since w_x is analytic in x, $w_x(x, t) \equiv 0$ so that $l(t) = -w_x(0, t) = 0$, which is impossible if $t \neq t_i$.)

The number of curves $a_i(t)$ is non-increasing in t; it may however decrease in t. From Lemma 3.2 we see that if $u(1, t)$ changes the direction of monotonicity at $t = t^*$, then on the right-most x-interval where $w_x(x, t) \neq 0$ the signature of w_x changes across $t = t^*$, so that the number of intervals (a_i, b_i) actually decreases as we cross $t = t^*$. We thus conclude:

LEMMA 3.3. *$w(1, t)$ is piecewise monotone; in fact, $w_t(1, t)$ can vanish at most at a finite number of points.*

LEMMA 3.4. *Let $\sigma_1 = \sup\{t; u(1, s) \geqslant 1$ for $0 \leqslant s \leqslant t\}$. Then for all $t > \sigma_1$, $t - \sigma_1$ small,*

$$w_x(x, t) > 0 \qquad \textit{if} \quad 1 - \delta(t) < x < 1$$

for some function $\delta(t) > 0$.

Proof. If the assertion is not true then either

$$w_x(x, \sigma_1) < 0 \qquad \text{if} \quad 1 - \delta < x < 1 \quad (\delta > 0),$$

or

$$w_x(x, \sigma_1) = 0 \qquad \text{if} \quad 1 - \delta < x < 1 \quad (\text{and then } l(\sigma_1) = 0)$$

but

$$w_x(x, t) < 0 \qquad \text{if} \quad 1 - \delta(t) < x < 1$$

for some $\delta(t) > 0$ and all $t > \sigma_1$, $t - \sigma_1$ small; this follows from the two paragraphs preceding Lemma 3.3. In both cases, Lemma 3.2 implies that $w(1, t)$ is increasing for $t > \sigma_1$, $t - \sigma_1$ small, thus contradicting the definition of σ_1.

Consider now the hydraulic problem: find $z(x, t)$, $s(t)$ satisfying:

$$\begin{aligned} z_t - z_{xx} &= 0 && \text{if} \quad 0 < x < 1, \quad t > \sigma_1, \\ z_x(x, 0) &= -l(t) && \text{if} \quad t > \sigma_1, \\ z(x, 0) &= w(x, \sigma_1) && \text{if} \quad 0 < x < 1, \\ z &= s && \text{if} \quad x = s(t), \quad t > \sigma_1, \\ z_x &= -\dot{s} && \text{if} \quad x = s(t), \quad t > \sigma_1. \end{aligned} \tag{3.5}$$

By [5], this problem has a unique solution as long as $s(t)$ remains positive. We claim that

$$s(t) > 0 \qquad \text{for all} \quad t > \sigma_1. \tag{3.6}$$

Suppose (3.6) is not true and $s(t^*) = 0$, $s(t) > 0$ if $\sigma_1 < t < t^*$. Integrating the relation $z_t - z_{xx} = 0$ over $0 < x < 1$, $\sigma_1 < t < t^*$, we obtain

$$\int_{\sigma_1}^{t^*} l(s)\, ds = -\tfrac{1}{2} - \int_0^1 z(x, \sigma_1)\, dx. \tag{3.7}$$

Similarly, integrating the relation $w_t - w_{xx} = 0$ over $0 < x < 1, 0 < t < \sigma_1$, we obtain an expression for

$$\int_0^{\sigma_1} l(s)\, ds.$$

Adding these two expressions, we find that

$$\int_0^{t^*} l(s)\, ds = -\tfrac{1}{2} - \frac{b^2}{2} - \int_0^x g(x)\, dx,$$

which contradicts (1.1), since $b > 1$.

We have thus proved that the solution of (3.5) exists for all $t > \sigma_1$. Set

$$\sigma_2 = \sup\{t; t \geqslant \sigma_1, s(\lambda) \leqslant 1 \text{ if } \sigma_1 \leqslant \lambda \leqslant t\}. \tag{3.8}$$

We claim that

$$\sigma_2 > \sigma_1 . \tag{3.9}$$

Indeed, from the definition of σ_1 it follows that either

$$w_x(x, \sigma_1) > 0 \text{ in some interval } 1 - \delta < x < 1$$

or at least

$$w_x(x, t) > 0 \quad \text{if} \quad 1 - \delta(t) < x < 1, \quad \delta(t) > 0$$

for all $t > \sigma_1$, $t - \sigma_1$ sufficiently small. In either case we can apply an argument as in Lemma 3.2 to deduce that $s(t)$ is strictly decreasing. Instead of a region R bounded by γ, I and $\tilde{\gamma}$ where $\tilde{\gamma}$ is a line segment on $x = 1$, we now define R in the same way except that $\tilde{\gamma}$ is a part of the free boundary.

For $t > \sigma_2$ we consider the problem

$$\begin{aligned}
w_t - w_{xx} &= 0 && \text{if} \quad 0 < x < 1, \quad t > \sigma_2 , \\
w_x(0, t) &= -l(t) && \text{if} \quad t > \sigma_2 , \\
w(x, \sigma_2) &= z(x, \sigma_2) && \text{if} \quad 0 < x < 1, \\
w_x + w w_t &= 0 && \text{if} \quad x = 1, \quad t > \sigma_2 .
\end{aligned} \tag{3.10}$$

Using the argument of Lemma 3.2 for z and then for w at $t = \sigma_2$ (or at $t > \sigma_2$, $t - \sigma_2$ small), we find that $w(1, t)$ is strictly increasing in some interval $\sigma_2 < t < \sigma_2 + \delta$. Let

$$\sigma_3 = \sup\{t; t > \sigma_2 , w(1, \lambda) \geqslant 1 \text{ if } \sigma_2 \leqslant \lambda \leqslant t\}.$$

Next we define $z(x, t)$ for $t > \sigma_3$ as the solution of (3.5) with σ_1 replaced by σ_3. We next define σ_4 analogously to σ_2, $w(x, t)$ for $t > \sigma_4$ analogously to (3.10), with σ_2 replaced by σ_4, etc. This process comes to an end with some σ_n, $n \leqslant N$, and then $\sigma_{n+1} = \infty$.

We define

$$\begin{aligned}
\tilde{u}(x, t) &= w(x, t) && \text{if} \quad 0 \leqslant t < \sigma_1 , \\
& z(x, t) && \text{if} \quad \sigma_1 \leqslant t < \sigma_2 , \\
& w(x, t) && \text{if} \quad \sigma_2 \leqslant t < \sigma_3 , \\
& \text{etc.,} && \\
\tilde{s}(t) &= \tilde{u}(1, t) && \text{if} \quad 0 \leqslant t < \sigma_1 , \\
& s(t) && \text{if} \quad \sigma_1 \leqslant t < \sigma_2 , \\
& \tilde{u}(1, t) && \text{if} \quad \sigma_2 \leqslant t < \sigma_3 , \\
& \text{etc.}
\end{aligned} \tag{3.11}$$

We can now state the main result of this section:

THEOREM 3.5. *For any* $T > 0$,

$$\lim_{k\to\infty} s^k(t) = \tilde{s}(t) \text{ uniformly in } t, \quad 0 \leqslant t \leqslant T, \tag{3.12}$$

and

$$\lim_{k\to\infty} u^k(x, t) = \tilde{u}(x, t) \tag{3.13}$$

uniformly in compact subsets of $0 \leqslant x < \tilde{s}(t)$, $0 \leqslant t < \infty$.

Proof. From Corollary 2.2 we obtain

$$| \dot{s}^k(t)| \leqslant C, \tag{3.14}$$

$$| u_x{}^k(x, t)| < \frac{C}{k} \qquad \text{if} \quad 1 < x < s^k(t), \tag{3.15}$$

$$| u_x{}^k(x, t)| \leqslant C \qquad \text{if} \quad 0 < x < \min(1, s^k(t)). \tag{3.16}$$

By integrating the equation $(ku_x{}^k)_x = u_t{}^k$ for $0 < x < s^k(\lambda)$, $0 < \lambda < t$, and using (1.1) we find that not only $s^k(t)$ remains positive but also that, for any $T > 0$,

$$s^k(t) \geqslant \epsilon_0 \qquad \text{for all} \quad k, 0 < t < T,$$

where ϵ_0 is some positive constant independent of k. We can then deduce the inequalities (cf. the derivation of (2.7)–(2.9))

$$| u_t{}^k(x, t)| \leqslant C \qquad \text{if} \quad \delta < x < s^k(t), \quad 0 < t < T, \tag{3.17}$$

$$| u_{xx}^k(x, t)| \leqslant C \qquad \text{if} \quad \delta < x < s^k(t), \quad 0 < t < T, \quad x \neq 1, \tag{3.18}$$

where δ is any positive number smaller than ϵ_0. By interpolation [6] we then also have

$$| u_x{}^k(x, t) - u_x{}^k(x, t')| \leqslant C \, | t - t' |^{1/2}, \qquad x \neq 1. \tag{3.19}$$

Take any sequence $\{k'\}$. Then we can extract a subsequence $\{k''\}$ such that

$$u^{k''} \to \bar{u}, \qquad s^{k''} \to \bar{s}$$

where the convergence is uniform in the same sense as in (3.12), (3.13). Furthermore,

$$\bar{u}_x(x, t) = 0 \qquad \text{if} \quad x > \bar{s}(t). \tag{3.20}$$

If we show that

$$\bar{s}(t) = \tilde{s}(t), \qquad \bar{u}(x, t) = \tilde{u}(x, t) \tag{3.21}$$

then the assertion of Theorem 3.5 follows.

We first verify (3.21) for all $t < \sigma_1$.

Let $\bar{\sigma}_1 = \sup\{t; \bar{s}(\lambda) \geqslant 1 \text{ if } \lambda \leqslant t\}$, and write, for simplicity, $k'' = k$. Then

$$s^k(t) \geqslant 1 - \epsilon_k \qquad \text{if} \quad 0 \leqslant t \leqslant \bar{\sigma}_1$$

where $\epsilon_k \to 0$. Consider an interval $(0, T)$ where $s^k(t) > 1$ for all $t \in (0, T)$ and k large. Integrating the relation $ku_{xx}^k = u_t^k$ over $1 < x < s^k(t)$, $0 < t < \lambda$ where $0 < \lambda < T$, then taking $k \to \infty$ and using (2.20) we obtain the relation

$$-\int_0^\lambda \bar{u}_x(1 - 0, t)\, dt = \tfrac{1}{2}\bar{s}^2(\lambda) - \tfrac{1}{2}b^2 - b.$$

Also, since $\bar{u}_x(x, \lambda) = 0$ if $x > 1$,

$$\bar{u}(1 - 0, \lambda) = \bar{u}(1 + 0, \lambda) = \bar{s}(\lambda).$$

It follows $\bar{s}(\lambda)$ is differentiable and

$$\bar{u}_x + \bar{u}\bar{u}_t = 0 \qquad \text{on} \quad x = 1 - 0. \tag{3.22}$$

Now $\dot{s}^k(t)$ can change sign at most N times; the proof for $k(x) \equiv 1$ is given in [5] and the proof for $k(x)$ piecewise constant is the same.

Denote by $t_1^k, t_2^k, \ldots$ the successive points where $s^k(t) = 1$. We may assume that $\lim t_i^k$ exists, and we denote it by τ_i. Then (3.22) is valid for all $t < \tau_1$. If $\tau_2 < \bar{\sigma}_1$ and $\tau_1 < \lambda < \mu < \tau_2$ then, for all k sufficiently large,

$$1 - \epsilon_k < s^k(t) < 1 \qquad \text{if} \quad \lambda \leqslant t \leqslant \mu \ (\epsilon_k \to 0 \text{ if } k \to \infty).$$

Hence,

$$\int_\lambda^\mu u_x^k(s^k(t) - 0, t)\, dt = s^k(\mu) - s^k(\lambda) \to 0$$

so that

$$\int_\lambda^\mu \bar{u}_x(1 - 0, t)\, dt = 0, \qquad \text{i.e., } \bar{u}_x(1 - 0, t) = 0.$$

On the other hand, $\bar{u}(1, t) = \lim u^k(s^k(t), t) = \lim s^k(t) = 1$ if $\tau_1 < t < \tau_2$. Hence $\bar{u}\bar{u}_t = 0$ if $x = 1$, $\tau_1 < t < \tau_2$. Thus the relation (3.22) holds in (τ_1, τ_2). We can now proceed to establish (3.22) for $\tau_2 < t < \tau_3$ (if $\tau_3 < \bar{\sigma}_1$) by the same method as for $(0, \tau_1)$, then for $\tau_3 < t < \tau_4$, etc. In this way we establish that $\bar{u}$ satisfies (3.1) if $t < \bar{\sigma}_1$. Since $u^k(x, t) > x$ if $0 < x < 1$, we must have $\bar{u}(1, t) \geqslant 1$, and therefore $\sigma_1 \geqslant \bar{\sigma}_1$. For $t > \bar{\sigma}_1$ the curve $x = \bar{s}(t)$ lies in $x < 1$ if $t - \bar{\sigma}_1$ is sufficiently small; here we use the fact that each s^k changes the direction of monotonicity a finite number of times, $\leqslant N$. However, it is easily seen that $\bar{u}, \bar{s}$ form a solution of the hydraulic problem for $t > \bar{\sigma}_1$, $t - \sigma_1$ small. Hence $\bar{\sigma}_1 = \sigma_1$. Since $(\tilde{u}, \tilde{s})$ is a solution of the same problem for $\sigma_1 < t < \sigma_2$, we deduce that $\bar{u} = \tilde{u}$, $\bar{s} = \tilde{s}$ for all $t < \sigma_2$, and $\sigma_2 \leqslant \bar{\sigma}_2$. We next prove that $\sigma_2 = \bar{\sigma}_2$ and that $\tilde{u} = \bar{u}$, $\tilde{s} = \bar{s}$ for all $\sigma_2 < t < \bar{\sigma}_3$, etc.

Remark. Denote the right hand side of (1.1) by γ_0. Since $g(x) > x$, $\gamma_0 < 0$. Let

$$\gamma = -\tfrac{1}{2}b^2 - \int_0^1 g(x)\,dx + 1.$$

Then $\gamma = \gamma_0 + 1$. Therefore we can construct a function $l(t)$ satisfying (1.1) such that

$$\int_0^{\tau_1} l(t)\,dt \leqslant \gamma \qquad \text{for some} \quad \tau_1 > 0. \tag{3.23}$$

We claim that then $\sigma_1 < \tau_1$ in Theorem 3.5. Indeed, otherwise we integrate $w_t - w_{xx} = 0$ over $0 < x < 1$, $0 < t < \tau_1$ and make use of the inequalities $w(x, t) > x$ (since $u^k(x, t) > x$) and $w(1, t) \geqslant 1$. We find that

$$\int_0^{\tau_1} l(t)\,dt > \gamma,$$

which contradicts (3.23).

Similarly one can show that if

$$\int_0^{\tau_2} l(t)\,dt > \gamma_1 \qquad \text{for some} \quad \tau_2\,,$$

where γ_1 depends on $\sup w(x, \sigma_1)$, then $\sigma_2 < \tau_2$. Assuming next that

$$\int_0^{\tau_3} l(t)\,dt < \gamma \qquad \text{for some} \quad \tau_3 > \sigma_2\,,$$

we again conclude that $\sigma_3 < \tau_3$, etc. In this way we can construct a function $l(t)$ whose number of oscillations in $(0, T)$ increases to ∞ as $T \to \infty$, such that the number of oscillations of the corresponding free boundary $\bar{s}(t)$ in $(0, T)$ also increases to ∞ as $T \to \infty$.

4. The Case $(k, 1)$, $k \to \infty$

Consider the problem: find $w(x, t)$, $s(t)$ satisfying:

$$\begin{aligned}
w_t - w_{xx} &= 0 && \text{if} \quad 1 < x < s(t), \quad t > 0,\\
w(x, 0) &= g(x) && \text{if} \quad 1 < x < b,\\
w_x &= w_t && \text{if} \quad x = 1, \quad t > 0,\\
w &= s(t) && \text{if} \quad x = s(t), \quad t > 0,\\
w_x &= -\dot{s}(t) && \text{if} \quad x = s(t), \quad t > 0.
\end{aligned} \tag{4.1}$$

This is the hydraulic problem considered earlier, except that the condition $w_x = -l(t)$ at the fixed left endpoint is replaced by the condition $w_x = w_t$ at the left endpoint.

THEOREM 4.1. *There exists a unique solution of* (4.1) *for all* $t > 0$.

Proof. As in the case of Theorem 3.1, we first consider the same problem for $v = w_x$, that is, defining

$$w(x, t) = \int_0^x v(y, t)\, dy + \int_0^t v_x(1, \tau)\, d\tau + g(0),$$

the problem (4.1) is equivalent to the problem

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{if } \ 1 < x < s(t), \quad t > 0, \\ v(x, 0) &= g'(x) && \text{if } \ 0 < x < 1, \\ v &= v_x && \text{if } \ x = 1, \quad t > 0, \\ v &= -\dot{s}(t) && \text{if } \ x = s(t), \quad t > 0, \\ v_x &= \dot{s}^2(t) + \dot{s}(t) && \text{if } \ x = s(t), \quad t > 0. \end{aligned} \tag{4.2}$$

We first establish a priori bounds on the solution, if existing. By applying the maximum principle to the function $w - x$ we find that it cannot take negative minimum on $x = 1$. Hence $w - x > 0$ if $0 < x < s(t)$, $t > 0$. It follows that

$$\frac{\partial}{\partial x}(w - x) < 0 \qquad \text{on} \quad x = s(t),$$

so that $1 + \dot{s}(t) > 0$. This can be used to deduce that w_x cannot take an extremum on the free boundary (see [5; p. 467]). It is also easily seen that w_x cannot attain an extremum on $x = 1$. It follows that

$$| w_x | \leqslant \sup | g'(x)|. \tag{4.3}$$

With this a priori bound at hand, we shall proceed to establish the existence of a unique solution of (4.2) by the method of integral equations (cf. [3] [4] [5]) as long as $s(t) - 1$ remains positive, say $s(t) - 1 \geqslant \epsilon$, $\epsilon > 0$; since the length of the time interval will depend only on ϵ, sup $|u_x(x, 0)|$, the assertion of Theorem 4.1 would then follow upon using (4.3), provided $s(t) - 1$ remains positive.

Let $N(x, y, t)$ be the Neumann function for the heat equation in the region $x > 1$, $t > 0$. Then we can express any possible solution v, s of (4.2) in the form:

$$v(x, t) = -\int_0^t vN\,|_{y=1} + \int_0^t vN_y\,|_{y=s(t)} - \int_0^t v_yN\,|_{y=s(t)} + \int_1^b g'N\, dy. \tag{4.4}$$

We now let $x \to s(t)$ and use the relations

$$v = -\dot{s}, \qquad v_y = \dot{s}^2 + \dot{s} \qquad \text{on} \quad y = s(t)$$

and a standard jump relation of $\int^t N_x \rho \,|_{x=s(t)}\, dt$ (see [4; p. 217]). We then obtain a nonlinear Volterra integral equation, which we write briefly as

$$\dot{s}(t) = I_1(t, s(\cdot), v(1, \cdot)). \tag{4.5}$$

Next, letting $x \to 1$ in (4.4) we obtain an integral equation

$$v(1, t) = I_2(t, s(\cdot), v(1, \cdot)). \tag{4.6}$$

By a standard method one shows (cf. [3]) that there exists a unique solution of (4.5), (4.6) in a small time interval and $s(t)$ is continuously differentiable.

Let $\tilde{v}$ be the solution of

$$\begin{aligned}
\tilde{v}_t - \tilde{v}_{xx} &= 0 && \text{if} \quad 1 < x < s(t), \quad t > 0,\\
\tilde{v}(x, 0) &= g'(x) && \text{if} \quad 0 < x < 1,\\
\tilde{v}_x(1, t) &= v(1, t) && \text{if} \quad t > 0,\\
\tilde{v}_x &= \dot{s}^2(t) + \dot{s}(t) && \text{if} \quad x = s(t), \quad t > 0.
\end{aligned}$$

We represent $\tilde{v}(x, t)$ by means of the Neumann function and let $x \to s(t)$. Comparing with (4.5), we obtain for $\tilde{v}(s(t), t) + \dot{s}(t)$ a homogeneous Volterra integral equation. It follows that $\tilde{v} = -\dot{s}(t)$ on $x = s(t)$. Next we let $x \to 1$ in the representation for $\tilde{v}(x, t)$. We obtain for $\tilde{v}(1, t)$ the same expression as for $v(1, t)$ in (4.6). Hence $\tilde{v}(1, t) = v(1, t)$. It follows that $\tilde{v}(x, t)$ is the solution of (4.2).

It remains to show that $s(t) - 1$ remains positive for all $t > 0$. If this is not true then there is a t_0 such that $s(t) > 1$ if $t < t_0$, $s(t_0 - 0) = 1$. Integrating $w_t - w_{xx} = 0$ over $\lambda < x < s(t)$, $0 < t < t_0 - \delta$ and taking $\delta \to 0$ we find, upon using the boundary conditions in (4.1), that

$$g(1) + \int_1^b g(x)\, dx = \frac{3}{2} + \frac{b^2}{2} - b.$$

But since $g(x) > x$, $g(1) > 1$, we obtain the inequality $b < 1$, which is impossible.

We can now state the main result of this section.

THEOREM 4.2. *For any* $T > 0$,

$$\lim_{k\to\infty} s^k(t) = s(t) \textit{ uniformly in } t, \quad 0 \leqslant t \leqslant T, \tag{4.7}$$

and

$$\lim_{x\to\infty} u^k(x, t) = w(x, t) \tag{4.8}$$

uniformly in compact subsets of $1 \leqslant x < s(t)$, $t \geqslant 0$, *where* (w, s) *is the unique solution of* (4.1).

Proof. As in the proof of Theorem 3.5 we take a subsequence $(u^{k''}, s^{k''})$, which we again denote by (u^k, s^k) such that

$$s^k \to \bar{s}, \qquad u^k \to \bar{u}$$

with suitable number of derivatives. We now have

$$\bar{u}_x(x, t) = 0 \qquad \text{if} \quad 0 < x < 1 < \bar{s}(t), \quad t > 0. \tag{4.9}$$

We claim that

$$\bar{s}(t) > 1 \qquad \text{for all} \quad t > 0. \tag{4.10}$$

Indeed, otherwise there is a first time $\bar{t}$ such that $s(t) > 1$ if $0 < t < \bar{t}$, $\bar{s}(\bar{t}) = 1$. Notice that

$$\begin{aligned} \bar{u}_t &= \bar{u}_{xx} && \text{if} \quad 1 < x < \bar{s}(t), \quad 0 < t < \bar{t}, \\ \bar{u} &= \bar{s}(t) && \text{if} \quad x = \bar{s}(t), \quad 0 < t < \bar{t}, \\ \bar{u}_x &= -\bar{s}(t) && \text{if} \quad x = \bar{s}(t), \quad 0 < t < \bar{t}, \\ \bar{u}(x, 0) &= g(x) && \text{if} \quad 1 < x < b. \end{aligned} \tag{4.11}$$

Finally, if we integrate the relation $ku_{xx}^k = u_t^k$ over $1 < x < s^k(t)$, $0 < t < \lambda$ where $\lambda < \bar{t}$ and k is sufficiently large, and then take $k \to \infty$, we obtain

$$\int_0^\lambda \bar{u}_x(1 + 0, t)\, dt = u(1, \lambda) - g(t).$$

Hence

$$\bar{u}_t = \bar{u}_x \quad \text{on} \quad x = 1, \quad 0 < t < \bar{t}. \tag{4.12}$$

Integrating the relation $\bar{u}_t = \bar{u}_{xx}$ over $1 < x < s(t)$, $0 < t < \bar{t}$ and using the boundary conditions of (4.11), (4.12) and $\bar{s}(\bar{t}) = 1$, we find that

$$g(1) + \int_1^1 g(x)\, dx = \frac{3}{2} + \frac{b^2}{2} - b,$$

which is impossible.

We have thus proved that (4.10) holds for all $t > 0$. At the same time we have also verified that $\bar{u}, \bar{s}$ form a solution of the hydraulic problem (4.1). This completes the proof of the theorem.

Remark. An indirect proof of the existence part of Theorem 4.1 follows from the above argument that $(\lim u^{k'}, \lim s^{k'})$ form such a solution.

5. The Case $(1, k_\nu 1)$, $k \to \infty$

As in Section 3 we shall impose in this section the condition (3.4).

Set $\gamma = \beta - 1$ and consider the free boundary problem: find functions $w(x, t)$, $s(t)$ satisfying

$$\begin{aligned}
w_t - w_{xx} &= 0 && \text{if } 0 < x < 1 \text{ or } \beta < x < s(t),\ t > 0,\\
w_x(0, t) &= -l(t) && \text{if } t > 0,\\
w(x, 0) &= g(x) && \text{if } 0 < x < 1 \text{ or } \beta < x < b,\\
w &= s(t) && \text{if } x = s(t),\ t > 0,\\
w_x &= -\dot{s}(t) && \text{if } x = s(t),\ t > 0,\\
w(\beta, t) &= w(1, t) && \text{if } t > 0,\\
w_x(\beta, t) - w_x(1, t) &= \gamma w_t(1, t) && \text{if } t > 0.
\end{aligned} \tag{5.1}$$

We are interested in a solution for which $s(t) \geqslant \beta$.

We claim that

$$\text{there exists a unique solution of (5.1) for all } t > 0, \text{ as long as } s(t) > \beta. \tag{5.2}$$

Indeed, first we prove an a priori bound on w_x. To do this let us first show that

$$w_x \text{ cannot take an extremum on } x = 1 \text{ or } x = \beta. \tag{5.3}$$

Suppose for instance that w_x, on $0 \leqslant x \leqslant 1$, $\beta \leqslant x \leqslant s(t)$, $0 \leqslant t \leqslant T$ takes minimum G at a point $x = \beta$, $t = t_0$. Then $w_{xx}(\beta, t_0) > 0$ so that $w_t(\beta, t_0) > 0$. It follows from the last equation in (5.1) that $w_x(1, t_0) < w_x(\beta, t_0) = G$, which is impossible. The proof that the minimum of w_x cannot be taken on $x = 1$ is similar.

Using (5.3) we can now apply the argument leading to the estimate of Corollary 2.6 in order to show that w_x cannot take an extremum on the free boundary. We thus conclude, by the maximum principle, that the a priori estimate

$$| w_x | \leqslant \max\{\sup | g' |, \sup | l |\}$$

is valid. Thus, in order to complete the proof of (5.2) it remains to prove a local existence and uniqueness theorem. This we can do by working with $v = w_x$ and deriving integral equations for $\dot{s}(t)$, $v(1, t)$, $v(\beta, t)$ (cf. the proof of Theorem 4.1).

We extend the solution w by setting $w(x, t) = w(1, t)$ if $1 < x < \beta$.

Let $\bar{t}_1$ be such that the solution of (5.1) exists (with $s(t) > \beta$) for all $t < \bar{t}_1$ and $s(\bar{t}_1) = \beta$. Consider the solution $\tilde{u}$, $\tilde{s}$ constructed in Section 3 (with $t = 0$, $g(x)$ replaced by $t = \bar{t}_1$, $w(x, \bar{t}_1)$).

LEMMA 5.1. *There exists a* $\delta > 0$ *such that either*

(a) $\bar{u}(1, t)$ *is strictly decreasing in* $\bar{t}_1 < t < \bar{t}_1 + \delta$, *and then* (5.1) *has no solution* (*with* w_x, $\dot{s}$ *bounded and* $s(t) > \beta$) *in any subinterval* $\bar{t}_1 < t < \bar{t}_1 + \bar{\delta}$ ($\bar{\delta} < \delta$); *or*

(b) $\bar{u}(1, t)$ *is strictly increasing in* $\bar{t}_1 < t < \bar{t}_1 + \delta$, *and then* (5.1) *has a unique solution* (*with* $w_x . \dot{s}$ *bounded*, $s(t) > 0$) *in* $\bar{t}_1 < t < t_1 + \delta$; *further* $s(t)$ *is monotone increasing.*

Proof. For simplicity we consider first the analogous case with $\bar{t}_1 = 0$. Thus we suppose that $s(0) = b = \beta$. Extend $g(x)$ by $g(x) = g(\beta)$ for $x > \beta$ and consider the "ϵ-problem" (5.1) whereby $s(0) = \beta + \epsilon$. The solution (w_ϵ, s_ϵ) exists for some time T_ϵ.

Suppose $g'(x) < 0$ if $1 - \delta_1 < x < 1$, for some $\delta_1 > 0$. Let $1 - \delta_1 < \theta < 1$ and consider (w_ϵ, s_ϵ) for $x > \theta$. If $T_\epsilon \leqslant \delta^*$ (where δ^* is a sufficiently small positive number, independent of ϵ) then $w_{\epsilon x} < 0$ if $x = \theta$, $0 < t < \delta^*$. Also $w_{\epsilon x} < 0$ on $t = 0$, $\theta < x < 1$. Recall that $w_{\epsilon x}$ cannot take its extremum on $x = 1$, $x = \beta$ or on the free boundary. It follows, by the maximum principle, that $w_{\epsilon x} < 0$ if $\theta < x < 1$ or $\beta < x < s_\epsilon(t)$ and $0 < t < T_\epsilon$. Hence $\dot{s}_\epsilon(t) > 0$, that is, $s_\epsilon(t)$ is strictly increasing. We can therefore continue the solution w_ϵ, s_ϵ beyond $t = T_\epsilon$, if $T_\epsilon < \delta^*$. This procedure can be repeated as long as $t < \delta^*$. Thus the solution (w_ϵ, s_ϵ) exists for all $t \leqslant \delta^*$. Taking $\epsilon \to 0$ we obtain a pair (w, s) such that either it is a solution of (5.1) for $t \leqslant \delta^*$ or $s(t) \equiv 0$. The latter situation cannot occur; for it would imply that $s_\epsilon(t) \to 1$ uniformly, which, in turn, would make w a solution of (3.1) with $w = w_x = 0$ on $x = 1$ (which is impossible). The uniqueness of the solution follows by the method of integral equations (which one can write for $\dot{s}(t)$, $v(\beta, t)$, $v(1, t)$, where $v = w_x$).

Consider next the case where $g'(x) > 0$ in some interval $1 - \delta_1 < x < 1$. We shall prove that (5.1) (with $s(0) = b = \beta$) does not have a solution. Indeed, if (w, s) is a solution then, for any $1 - \delta < \theta < 1$, (w, s) is also a solution for $x > \theta$, and $w_x > 0$ if $x = \theta$, $0 < t < \delta^*$, or if $t = 0$, $\theta < x < 1$, where δ^* is positive and sufficiently small. Since, as before, w_x cannot take an extremum on $x = 1$, $x = \beta$ or on the free boundary, we deduce that $w_x > 0$ if $x > \theta$. In particular, $\dot{s}(t) < 0$, which is impossible.

If $g'(x) = 0$ in some interval $1 - \delta_1 < x < 1$, we can repeat the above argument by looking at the sign of $g'(x)$ in the right-most subinterval $x_1 < x < x_2$ of (0, 1) where $g'(x) \neq 0$.

Note that if $g'(x) > 0$ (< 0) in $1 - \delta_1 < x < 1$ or in the right-most interval (in (0, 1)) where $g'(x) \neq 1$, then $\bar{u}(1, t)$ is strictly decreasing (increasing) in some interval $0 < t < \delta$. We have thus proved the lemma in case $\bar{t}_1 = 0$. The proof for $\bar{t}_1 > 0$ is similar.

We now check whether, at $t = \bar{t}_1$, (a) or (b) holds. If (a) holds then we set $\hat{\sigma}_1 = \bar{t}_1$. If (b) holds, we define $\bar{t}_2$ to be the smallest number such that

$$\bar{t}_2 > \bar{t}_1, \qquad s(t) > \beta \qquad \text{if} \quad \bar{t}_1 < t < \bar{t}_2, \qquad s(\bar{t}_2) = 0.$$

If case (a) holds at $t = \bar{t}_2$, then we define $\hat{\sigma}_1 = \bar{t}_2$; otherwise we define $\bar{t}_3$ in the obvious way, etc.

Recall that in this section we assume the condition (3.4). Therefore s^k can oscillate at most N times. The same is true of $\lim s^k$, which (as in the proof (4.7)) coincides with $s(t)$ (cf. (5.1)) as long as $t < \bar{t}_j$, for any j. It follows that, for some $j \leqslant N$, the case (a) must occur for $t = \bar{t}_j$. We then define $\hat{\sigma}_1 = \bar{t}_j$.

For $t > \hat{\sigma}_1$, the solution $\tilde{u}, \tilde{s}$ constructed in Section 3 (with $t = 0$, $g(x)$ replaced by $t = \hat{\sigma}_1$, $w(x, \hat{\sigma}_1)$) is such that $\tilde{u}(1, t)$ is strictly decreasing for some time interval (by Lemma 5.1).

Let $\hat{\sigma}_2$ be the first time such that

$$\begin{gathered} \hat{\sigma}_2 > \hat{\sigma}_1, \qquad \tilde{s}(t) \leqslant \beta \qquad \text{if} \quad \hat{\sigma}_1 < t < \hat{\sigma}_2, \qquad \tilde{s}(\hat{\sigma}_2) = \beta, \\ \tilde{u}(1, t) \text{ is strictly increasing in some interval } \hat{\sigma}_2 < t < \hat{\sigma}_2 + \delta_0 . \end{gathered} \tag{5.4}$$

For $t > \hat{\sigma}_2$ we can again construct a solution (w, s) of (5.1) (with $t = 0$, $g(x)$ replaced by $t = \hat{\sigma}_2$, $\tilde{u}(x, \hat{\sigma}_2)$). We then define $\hat{\sigma}_3$ analogously to $\hat{\sigma}_1$, then define $\tilde{u}, \tilde{s}$ and $\hat{\sigma}\bar{s}$, etc.

Denote by $(\tilde{u}(x, t), s(t))$ the pair of functions thus constructed.

THEOREM 5.2. *For any* $T > 0$,

$$\lim_{k \to \infty} s^k(t) = \hat{s}(t) \textit{ uniformly in } t, 0 \leqslant t \leqslant T, \tag{5.5}$$

and

$$\lim_{x \to \infty} u^k(x, t) = \hat{u}(x, t) \tag{5.6}$$

uniformly in compact sets of $0 \leqslant x < s(t)$, $t \geqslant 0$.

The proof exploits the arguments given in Sections 3, 4; it is omitted.

REFERENCES

1. L. A. CAFFARELLI AND A. FRIEDMAN, The dam problem with two layers, *Arch. Rational Mech. Anal.* **68** (1978), 125–154.
2. L. A. CAFFARELLI AND A. FRIEDMAN, Asymptotic estimates for the dam problem with several layers, *Indiana Univ. Math. J.* **27** (1978), 551–580.
3. A. FRIEDMAN, Free boundary problems for parabolic equations. I. Melting of solids, *J. Math. Mech.* **8** (1959), 499–518.
4. A. FRIEDMAN, "Partial Differential Equations of Parabolic Type," Prentice–Hall, Englewood Cliffs, N. J., 1964.
5. A. FRIEDMAN AND R. JENSEN, A parabolic quasi-variational inequality arising in hydraulics, *Ann. Scuola Norm. Sup. Pisa* (4) **2** (1975), 421–468.
6. R. JENSEN, Finite difference approximations to the free boundary of a parabolic variational inequality, to appear.
7. A. TORELLI, Existence and uniqueness of the solution for a non-steady free boundary problem, to appear.